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Cofree coalgebras over operads

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Abstract

This paper explicitly constructs cofree coalgebras over operads in the category of DG-modules. Special cases are also considered in which the general expression simplifies (such as the pointed, irreducible case).

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1. Introduction

We begin with the definition of the object of this paper:

Definition 1.1. Let R be a commutative ring with unit and C be an R-module. Then a coalgebra G will be called *the cofree coalgebra cogenerated by C* if

(1) there exists a morphism of R-modules

$$\varepsilon : G \to C$$

called the *cogeneration map*,

(2) given any coalgebra D and any morphism of R-modules

$$f : D \to C$$

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there exists a *unique* morphism of coalgebras

$$\hat{f}: D \to G$$

that makes the diagram

$$\begin{tikzcd} D \arrow[r, "\hat{f}"] \arrow[dr, "f"'] & G \arrow[d, "\varepsilon"] \\ & C \end{tikzcd}$$

commute.

If $\mathcal{V}$ is an operad (see Definition 2.7) and C is a R-free DG-module, then the same definition holds for coalgebras and coalgebra-morphisms (see Definition 2.7) over $\mathcal{V}$.

Remark 1.2. If they exist, it is not hard to see that cofree coalgebras must be *unique* up to an isomorphism.

Constructions of free *algebras* satisfying various conditions (associativity, etc.) have been known for many years: One forms a general algebraic structure implementing a suitable "product" and forms the quotient by a sub-object representing the conditions. Then one shows that these free algebras map to any other algebra satisfying the conditions. For instance, it is well-known how to construct the free *algebra* over an operad—see [6].

The construction of cofree coalgebras is dual to this, although Thomas Fox showed (see [1,2]) that they are considerably more complex than free algebras. Definition 1.1 implies that a cofree coalgebra cogenerated by a R-module, C, must contain isomorphic images of *all possible* coalgebras over C.

Operads (in the category of graded groups) can be regarded as "systems of indices" for parametrizing operations. They provide a uniform framework for describing many classes of algebraic objects, from associative algebras and coalgebras to Lie algebras and coalgebras.

In recent years, there have been applications of operads to quantum mechanics and homotopy theory. For instance, Steenrod operations on the chain-complex of a space can be codified by making this chain-complex a coalgebra over a suitable operad.

The definitive references on cofree coalgebras are the book [10] and two papers of Fox. Sweedler approached cofree coalgebras as a kind of dual of free algebras, while Fox studied them *ab initio*, under the most general possible conditions.

In Section 3, we describe the cofree coalgebra over an operad and prove that it has the required properties. Theorem 3.8 gives our result.

In Section 4 we consider special cases such as the pointed irreducible case in which the coproduct is dual to the operad compositions—see 4.10 and 4.14.

2. Operads

2.1. Notation and conventions

Throughout this paper, R will denote a commutative ring with unit. All tensor-products will be over R so that $\otimes = \otimes_R$.

Definition 2.1. Let C and D be graded R-modules. A map of graded modules $f: C_i \to D_{i+j}$ will be said to be of degree j.

Remark 2.2. For instance, the *differential* of a DG-module will be regarded as a degree -1 map.

We will make extensive use of the Koszul Convention (see [5]) regarding signs in homological calculations:

Definition 2.3. If $f: C_1 \to D_1$, $g: C_2 \to D_2$ are maps, and $a \otimes b \in C_1 \otimes C_2$ (where a is a homogeneous element), then $(f \otimes g)(a \otimes b)$ is defined to be $(-1)^{\deg(g)\cdot\deg(a)} f(a) \otimes g(b)$.

Remark 2.4. This convention simplifies many of the common expressions that occur in homological algebra—in particular it eliminates complicated signs that occur in these expressions. For instance, the differential, $\partial_\otimes$, of the tensor product $C \otimes D$ is $\partial_C \otimes 1 + 1 \otimes \partial_D$.

If f_i, g_i are maps, it is not hard to verify that the Koszul convention implies that $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2)\cdot\deg(g_1)}(f_1 \circ f_2 \otimes g_1 \circ g_2)$.

Another convention that we will follow extensively is tensor products, direct products, etc. are of *graded modules.*

Powers of DG-modules over R, such as C^n will be regarded as iterated R-tensor products:

$$C^n = \underbrace{C \otimes_R \cdots \otimes_R C}_{n\text{ factors}}.$$

2.2. Definitions

Before we can define operads, we need the following:

Definition 2.5. Let $\sigma \in S_n$ be an element of the symmetric group and let $\{k_1, \ldots, k_n\}$ be n nonnegative integers with $K = \sum_{i=1}^n k_i$. Then $T_{k_1,\ldots,k_n}(\sigma)$ is defined to be the element $\tau \in S_K$ that permutes the n blocks

$$(1, \ldots, k_1), (k_1 + 1, \ldots, k_1 + k_2) \cdots (K - K_{n-1}, \ldots, K)$$

as σ permutes the set $\{1, \ldots, n\}$.

Remark 2.6. Note that it is possible for one of the k's to be 0, in which case the corresponding block is empty.

The standard definition (see [6]) of an operad in the category of DG-modules is:

Definition 2.7. A sequence of differential graded R-free modules, $\{\mathcal{V}_i\}$, will be said to form a *DG-operad* if they satisfy the following conditions:

(1) there exists a *unit map* (defined by the commutative diagrams below)

$$\eta : R \to \mathcal{V}_1;$$

(2) for all $i > 1$, $\mathcal{V}_i$ is equipped with a left action of S_i, the symmetric group;
(3) for all $k \geqslant 1$, and $i_s \geqslant 0$ there are maps

$$\gamma : \mathcal{V}_k \otimes \mathcal{V}_{i_1} \otimes \cdots \otimes \mathcal{V}_{i_k} \otimes \to \mathcal{V}_i,$$

where $i = \sum_{j=1}^{k} i_j$.
The γ-maps must satisfy the following conditions:

Associativity: the following diagrams commute, where $\sum j_t = j$, $\sum i_s = i$, and $g_\alpha = \sum_{\ell=1}^{\alpha} j_\ell$ and $h_s = \sum_{\beta=g_{s-1}+1}^{g_s} i_\beta$:

$$\begin{tikzcd}
\mathcal{V}_k \otimes (\bigotimes_{s=1}^k \mathcal{V}_{j_s}) \otimes (\bigotimes_{t=1}^j \mathcal{V}_{i_t}) \arrow[r, "\gamma\otimes\mathrm{Id}"] \arrow[dd, "\text{shuffle}"'] & \mathcal{V}_j \otimes (\bigotimes_{t=1}^j \mathcal{V}_{i_t}) \arrow[d, "\gamma"] \\
 & \mathcal{V}_i \\
\mathcal{V}_k \otimes (\bigotimes_{t=1}^k \mathcal{V}_{j_t} \otimes (\bigotimes_{q=1}^{j_t} \mathcal{V}_{i_{g_{t-1}+q}})) \arrow[r, "\mathrm{Id}\otimes(\otimes_t\gamma)"'] & \mathcal{V}_k \otimes (\bigotimes_{t=1}^k \mathcal{V}_{h_t}) \arrow[u, "\gamma"']
\end{tikzcd} \tag{2.1}$$

Units: the following diagrams commute:

$$\begin{tikzcd}
\mathcal{V}_k \otimes \mathbb{Z}^k \arrow[r, "\cong"] \arrow[d, "\mathrm{Id}\otimes\eta^k"'] & \mathcal{V}_k \\
\mathcal{V}_k \otimes \mathcal{V}_1{}^k \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \otimes \mathcal{V}_k \arrow[r, "\cong"] \arrow[d, "\eta\otimes\mathrm{Id}"'] & \mathcal{V}_k \\
\mathcal{V}_1 \otimes \mathcal{V}_k \arrow[ur, "\gamma"'] &
\end{tikzcd} \tag{2.2}$$

Equivariance: the following diagrams commute:

$$\begin{tikzcd}
\mathcal{V}_k \otimes \mathcal{V}_{j_1} \otimes \cdots \otimes \mathcal{V}_{j_k} \arrow[r, "\gamma"] \arrow[d, "\sigma\otimes\sigma^{-1}"'] & \mathcal{V}_j \arrow[d, "\mathrm{T}_{j_1,\ldots,j_k}(\sigma)"] \\
\mathcal{V}_k \otimes \mathcal{V}_{j_{\sigma(1)}} \otimes \cdots \otimes \mathcal{V}_{j_{\sigma(k)}} \arrow[r, "\gamma"'] & \mathcal{V}_j
\end{tikzcd} \tag{2.3}$$

where $\sigma \in S_k$, and the σ^{-1} on the left permutes the factors $\{\mathcal{V}_{j_i}\}$ and the σ on the right simply acts on $\mathcal{V}_k$. See 2.5 for a definition of $\mathrm{T}_{j_1,\dots,j_k}(\sigma)$.

$$\begin{tikzcd} \mathcal{V}_k \otimes \mathcal{V}_{j_1} \otimes \cdots \otimes \mathcal{V}_{j_k} \arrow[r, "\gamma"] \arrow[d, "\mathrm{Id}\otimes\tau_1\otimes\cdots\tau_k"'] & \mathcal{V}_j \arrow[d, "\tau_1\oplus\cdots\oplus\tau_k"] \\ \mathcal{V}_k \otimes \mathcal{V}_{j_{\sigma(1)}} \otimes \cdots \otimes \mathcal{V}_{j_{\sigma(k)}} \arrow[r, "\gamma"'] & \mathcal{V}_j \end{tikzcd}$$

where $\tau_s \in S_{j_s}$ and $\tau_1 \oplus \cdots \oplus \tau_k \in S_j$ is the block sum.

The individual $\mathcal{V}_n$ that make up the operad $\mathcal{V}$ will be called its *components.*

For reasons that will become clear in the sequel, we follow the nonstandard convention of using *subscripts* to denote components of an operad—so $\mathcal{V} = \{\mathcal{V}_n\}$ rather than $\{\mathcal{V}(n)\}$. Where there is any possibility of confusion with grading of a graded groups, we will include a remark.

We will also use the term *operad* for DG-operad throughout this paper.

Definition 2.8. An operad, $\mathcal{V}$, is called *unital* if $\mathcal{V}$ has a 0-component $\mathcal{V}_0 = R$, concentrated in dimension 0 and augmentations

$$\varepsilon_n : \mathcal{V}_n \otimes \mathcal{V}_0 \otimes \cdots \otimes \mathcal{V}_0 = \mathcal{V}_n \to \mathcal{V}_0 = R$$

induced by their structure maps.

Remark 2.9. The literature contains varying definitions of the terms discussed here.

Our definition of unital and non-unital operad corresponds to that in [6]. On the other hand, in [7] Markl defines a *unital* operad to have a *unit* (i.e., the map $\eta : R \to \mathcal{V}_1$) and calls operads meeting the condition in Definition 2.8 *augmented unital*. None of Markl's operads have a 0-component and his definition of augmentation *only* involves the 1-component (so that the "higher" augmentation maps $\varepsilon_n : \mathcal{V}_n \to R$ do not have to exist).

2.3. The composition–representation

Describing an operad via the γ-maps and the diagrams in 2.7 is known as the γ-representation of the operad. We will present another method for describing operads more suited to the constructions to follow:

Definition 2.10. Let $\mathcal{V}$ be an operad as defined in 2.7, let n, m be positive integers and let $1 \leqslant i \leqslant n$. Define

$$\circ_i : \mathcal{V}_n \otimes \mathcal{V}_m \to \mathcal{V}_{n+m-1}$$

the ith *composition operation* on $\mathcal{V}$, to be the composite

$$\begin{tikzcd}
\mathcal{V}_n \otimes \mathcal{V}_m \arrow[d, equal] \\
\mathcal{V}_n \otimes R^{i-1} \otimes \mathcal{V}_m \otimes R^{n-i} \arrow[d, "1\otimes\eta^{i-1}\otimes 1\otimes\eta^{n-i}"] \\
\mathcal{V}_n \otimes {\mathcal{V}_1}^{i-1} \otimes \mathcal{V}_m \otimes {\mathcal{V}_1}^{n-i} \arrow[d, "\gamma"] \\
\mathcal{V}_{n+m-1}
\end{tikzcd}$$

The γ-maps defined in 2.7 and the composition-operations uniquely determine each other.

Definition 2.11. Let $\mathcal{V}$ be an operad, let $1 \leqslant j \leqslant n$, and let $\{\alpha_1, \ldots, \alpha_j\}$ be positive integers. Then define

$$L_j : \mathcal{V}_n \otimes \mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_j} \to \mathcal{V}_{n-j+\sum \alpha_i}$$

to be the composite

$$\begin{tikzcd}
\mathcal{V}_n \otimes \mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_j} \arrow[d, equal] \\
\mathcal{V}_n \otimes (\mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_j} \otimes R \otimes \cdots \otimes R) \arrow[d, "1\otimes(1^j\otimes\eta^{n-j})"] \\
\mathcal{V}_n \otimes (\mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_j} \otimes \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_1) \arrow[d, "\gamma"] \\
\mathcal{V}_{n+\sum_{i=1}^{j}(\alpha_i-1)}
\end{tikzcd} \tag{2.4}$$

Remark 2.12. Clearly, under the hypotheses above, $L_n = \gamma$.

Operads were originally called *composition algebras* and defined in terms of these operations—see [3].

Proposition 2.13. *Under the hypotheses of* 2.11, *suppose* $j < n$. *Then*

$$L_{j+1} = L_j \circ (* \circ_{j+1+\sum_{i=1}^{j} \alpha_i} *) : \mathcal{V}_n \otimes \mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_{j+1}} \to \mathcal{V}_{n+\sum_{i=1}^{j+1}(\alpha_i-1)}.$$

In particular, the γ*-map can be expressed as an iterated sequence of compositions and* γ*-maps and the composition-operations determine each other.*

Remark 2.14. We will find the compositions more useful than the γ-maps in studying algebraic properties of coalgebras over $\mathcal{V}$.

The map γ and the composition-operations $\{\circ_i\}$ define the γ- and the *composition–representations* of $\mathcal{V}$, respectively.

Proof. This follows by induction on j: it follows from the definition of the $\{\circ_i\}$ in the case where $j = 1$. In the general case, it follows by applying the associativity identities and the identities involving the unit map, $\eta: R \to \mathcal{V}_1$. Consider the diagram

$$\begin{array}{c}
\mathcal{V}_n \otimes (\mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_j} \otimes R^{n-j}) \otimes R^{j+\sum_{i=1}^{j}(\alpha_i-1)} \otimes \mathcal{V}_{\alpha_{j+1}} \otimes R^{n-j-1} \\
\Big\downarrow {\scriptstyle 1\otimes(1^j\otimes\eta^{n-j})\otimes\eta^{j+\sum_{i=1}^{j}(\alpha_i-1)}\otimes 1\otimes\eta^{n-j-1}} \\
\mathcal{V}_n \otimes (\mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_t} \otimes \mathcal{V}_1^{n-j}) \otimes \mathcal{V}_1^{j+\sum_{i=1}^{j}(\alpha_i-1)} \otimes \mathcal{V}_{\alpha_{j+1}} \otimes \mathcal{V}_1^{n-j-1} \\
\Big\downarrow {\scriptstyle \gamma\otimes 1^{n+\sum_{i=1}^{j}(\alpha_i-1)}} \\
\mathcal{V}_{n+\sum_{i=1}^{j}(\alpha_i-1)} \otimes (\mathcal{V}_1^{j+\sum_{i=1}^{j}(\alpha_i-1)} \otimes \mathcal{V}_{\alpha_{j+1}} \otimes \mathcal{V}_1^{n-j-1}) \\
\Big\downarrow {\scriptstyle \gamma} \\
\mathcal{V}_{n+\sum_{i=1}^{j+1}(\alpha_i-1)}
\end{array} \tag{2.5}$$

The associativity condition implies that we can shuffle copies of $\mathcal{V}_1$ to the immediate left of the rightmost term, and shuffle the $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_1$ on the right to get a factor on the left of

$$\mathcal{V}_{\alpha_1} \otimes \mathcal{V}_1^{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_j} \otimes \mathcal{V}_1^{\alpha_j}$$

and one on the right of

$$\mathcal{V}_1 \otimes \mathcal{V}_{\alpha_{j+1}}$$

(this factor of $\mathcal{V}_1$ exists because $j < n$) and we can evaluate γ on each of these before evaluating γ on their tensor product. The conclusion follows from the fact that each copy of $\mathcal{V}_1$ that appears in the result has been composed with the unit map η so the left factor is

$$\gamma\big(\mathcal{V}_{\alpha_1} \otimes \mathcal{V}_1^{\alpha_1}\big) \otimes \cdots \otimes \gamma\big(\mathcal{V}_{\alpha_j} \otimes \mathcal{V}_1^{\alpha_j}\big) = \mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_j}$$

and the right factor is

$$\gamma(\mathcal{V}_1 \otimes \mathcal{V}_{\alpha_{j+1}}) = \mathcal{V}_{\alpha_{j+1}}$$

so the entire expression becomes

$$\gamma\big(\mathcal{V}_n \otimes \mathcal{V}_{\alpha_1} \otimes \cdots \otimes \mathcal{V}_{\alpha_{j+1}} \otimes \mathcal{V}_1^{n-j-1}\big)$$

which is what we wanted to prove. □

The composition representation is complete when one notes that the various diagrams in 2.7 translate into the following relations (whose proof is left as an exercise to the reader):

Lemma 2.15. *Compositions obey the identities*

$$(a \circ_i b) \circ_j c = \begin{cases} (-1)^{\dim b \cdot \dim c} (a \circ_{j-n+1} c) \circ_i b & \text{if } i+n-1 \leqslant j, \\ a \circ_i (b \circ_{j-i+1} c) & \text{if } i \leqslant j < i+n-1, \\ (-1)^{\dim b \cdot \dim c} (a \circ_j c) \circ_{i+m-1} b & \text{if } 1 \leqslant j < i, \end{cases}$$

where $\deg c = m$, $\deg a = n$, *and*

$$a \circ_{\sigma(i)} (\sigma \cdot b) = \mathrm{T}_{\underbrace{1,\dots,n,\dots,1}_{i\text{th position}}}(\sigma) \cdot (a \circ_i b). \tag{2.6}$$

Given compositions, we define *generalized structure maps* of operads.

Definition 2.16. Let $\mathcal{V}$ be an operad and let $\mathbf{u} = \{u_1, \dots, u_m\}$, be a list of symbols, each of which is either a positive integer or the symbol $\bullet$. We define the *generalized composition* with respect to $\mathbf{u}$, denoted $\gamma_{\mathbf{u}}$, by

$$\gamma_{\mathbf{u}} = \gamma \circ \bigotimes_{j=1}^{m} \iota_j : \mathcal{V}_m \otimes \mathcal{V}_{u_1} \otimes \cdots \otimes \mathcal{V}_{u_m} \to \mathcal{V}_n,$$

where

$$n = \sum_{j=1}^{m} u_j$$

and we follow the convention that

(1) $\bullet = 1$ when used in a numeric context,
(2) $\mathcal{V}_\bullet = R$,
(3) $\iota_j = \begin{cases} 1 : \mathcal{V}_{u_j} \to \mathcal{V}_{u_j} & \text{if } u_j \neq \bullet, \\ \eta : R \to \mathcal{V}_1 & \text{otherwise.} \end{cases}$

Remark 2.17. If $\{u_{k_1}, \dots, u_{k_t}\} \subset \{u_1, \dots, u_m\}$ is the sublist of non-$\bullet$ elements, then $\gamma_{\mathbf{u}}$ is a map

$$\gamma_{\mathbf{u}} : \mathcal{V}_m \otimes \mathcal{V}_{k_1} \otimes \cdots \otimes \mathcal{V}_{k_t} \to \mathcal{V}_n.$$

Lemma 2.18. *Let* $\mathcal{V}$ *be an operad, let* $n, m, \alpha > 0$ *let* $1 \leqslant i \leqslant n$ *be integers, and let* $\mathbf{u} = \{u_1, \dots, u_n\}$, $\mathbf{v} = \{v_1, \dots, v_m\}$, $\mathbf{w} = \{w_1, \dots, w_{n+m-1}\}$ *be lists of symbols as in Definition* 2.16 *with*

$$\begin{aligned} &u_i \neq \bullet, \\ &u_i = \sum_{j=1}^{m} v_j, \\ &w_j = u_j \quad \text{if } j < i, \\ &w_j = v_{j-i+1} \quad \text{if } j \geqslant i \text{ and } j < i+m, \end{aligned}$$

$$
\begin{aligned}
w_j &= u_{j-m+1} \quad \textit{if } j \geqslant i+m, \\
\alpha &= \sum_{j=1}^{n+m-1} w_j \\
&= \sum_{j=1}^{n} u_j.
\end{aligned}
$$

Then the diagram

$$
\begin{tikzcd}
\mathcal{V}_n \otimes \mathcal{V}_m \otimes \bigotimes_{k=1}^{n+m-1} \mathcal{V}_{w_k} \arrow[r, "\circ_i \otimes 1"] \arrow[d, "1^{i-1} \otimes \gamma_{\mathbf{v}} \otimes 1^{n-i} \circ s"'] & \mathcal{V}_{n+m-1} \otimes \bigotimes_{k=1}^{n+m-1} \mathcal{V}_{w_k} \arrow[d, "\gamma_{\mathbf{w}}"] \\
\mathcal{V}_n \otimes \bigotimes_{k=1}^{n} \mathcal{V}_{u_k} \arrow[r, "\gamma_{\mathbf{u}}"'] & \mathcal{V}_\alpha
\end{tikzcd}
$$

commutes, where s is the shuffle map that sends $\mathcal{V}_m$ $i-1$ places to the right.

Remark 2.19. The conditions on $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ imply that $\mathbf{w}$ is the result of replacing u_i with the entire list $\mathbf{v}$.

Proof. This is a straightforward consequence of Definition 2.16 and the associativity condition in diagram (2.1). □

Morphisms of operads are defined in the obvious way:

Definition 2.20. Given two operads $\mathcal{V}$ and $\mathcal{W}$, a *morphism*

$$f : \mathcal{V} \to \mathcal{W}$$

is a sequence of chain-maps

$$f_i : \mathcal{V}_i \to \mathcal{W}_i$$

commuting with all the diagrams in 2.7 or (equivalently) preserving the composition operations in 2.16.

Now we give some examples:

Definition 2.21. The operad $\mathfrak{S}_0$ is defined via

(1) Its nth component is n—a chain-complex concentrated in dimension 0.
(2) Its structure map is given by

$$\gamma(1_{S_n} \otimes 1_{S_{k_1}} \otimes \cdots \otimes 1_{S_{k_n}}) = 1_{S_K},$$

where $1_{S_j} \in S_j$ is the identity element and $K = \sum_{j=1}^{n} k_j$. This definition is extended to other values in the symmetric groups via the equivariance conditions in 2.7.

Remark 2.22. This was denoted $\mathcal{M}$ in [6].

Verification that this satisfies the required identities is left to the reader as an exercise.

Definition 2.23. Let $\mathfrak{S}$ denote the operad with components $K(S_n, 1)$—the bar resolutions of $\mathbb{Z}$ over n for all $n > 0$. See [9] for formulas for the composition-operations.

Now we define an important operad associated to any R-module.

Definition 2.24. Let C be an R-free DG-module. Then the *Coendomorphism* operad, $\mathrm{CoEnd}(C)$, is defined to be the operad with component of rank $i = \mathrm{Hom}_R(C, C^i)$, with the differential induced by that of C and C^i. The dimension of an element of $\mathrm{Hom}_R(C, C^i)$ (for some i) is defined to be its degree as a map. If C is equipped with an augmentation

$$\varepsilon : C \to R,$$

where R is concentrated in dimension 0, then $\mathrm{CoEnd}(C)$ is unital, with 0 component generated by ε (with the identification $C^0 = R$).

Remark 2.25. One motivation for operads is that they model the iterated coproducts that occur in $\mathrm{CoEnd}(*)$. We will use operads as an algebraic framework for defining other constructs that have topological applications.

2.4. Coalgebras over an operad

Definition 2.26. Let $\mathcal{V}$ be an operad and let C be an R-free DG-module equipped with a morphism (of operads)

$$f : \mathcal{V} \to \mathrm{CoEnd}(C).$$

Then C is called a *coalgebra* over $\mathcal{V}$ *with structure map* f.

Remark 2.27. A coalgebra, C, over an operad, $\mathcal{V}$, is a sequence of maps

$$f_n : \mathcal{V} \otimes C \to C^n,$$

for all $n > 0$, where f_n is RS_n-equivariant or maps (via the *adjoint representation*):

$$g_n : C \to \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, C^n\big).$$

This latter description of coalgebras (via adjoint maps) is frequently more useful for our purposes than the previous one. In the case where $\mathcal{V}$ is *unital*, we write

$$\mathrm{Hom}_{RS_0}\big(\mathcal{V}_0, C^0\big) = R$$

and identify the adjoint structure map with the augmentation of C

$$g_0 = \varepsilon : C \to R = \mathrm{Hom}_{RS_0}\big(\mathcal{V}_0, C^0\big).$$

These adjoint maps are related in the sense that they fit into commutative diagrams:

$$\begin{tikzcd} C \arrow[r, "g_n"] \arrow[dd, "g_{n+m-1}"'] & \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^n) \arrow[d, "\mathrm{Hom}_R(1, \underbrace{1\otimes\cdots\otimes g_m\otimes\cdots\otimes 1}_{i\text{th position}})"] \\ & \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^{i-1}\otimes \mathrm{Hom}_{RS_m}(\mathcal{V}_m, C^m)\otimes C^{n-i}) \arrow[d, "\iota"] \\ \mathrm{Hom}_{RS_{n+m-1}}(\mathcal{V}_{n+m-1}, C^{n+m-1}) \arrow[r, "\mathrm{Hom}_R(\circ_i, 1)"'] & \mathrm{Hom}_R(\mathcal{V}_n\otimes\mathcal{V}_m, C^{n+m-1}) \end{tikzcd} \tag{2.7}$$

for all $m, n > 0$ and all $1 \leqslant i \leqslant n$, where ι is the composite

$$\begin{array}{c} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^{i-1}\otimes \mathrm{Hom}_{RS_m}(\mathcal{V}_m, C^m)\otimes C^{n-i}) \\ \| \\ \mathrm{Hom}_{RS_n}(\mathcal{V}_n, \mathrm{Hom}_R(R, C^{i-1})\otimes \mathrm{Hom}_{RS_m}(\mathcal{V}_m, C^m)\otimes \mathrm{Hom}_R(R, C^{n-i})) \\ \| \\ \mathrm{Hom}_R(\mathcal{V}_n, \mathrm{Hom}_R(\mathcal{V}_m, C^{m+n-1})) \\ \big\downarrow \\ \mathrm{Hom}_R(\mathcal{V}_n\otimes\mathcal{V}_m, C^{n+m-1}) \end{array} \tag{2.8}$$

In other words: The abstract composition-operations in $\mathcal{V}$ exactly correspond to compositions of maps in $\{\mathrm{Hom}_R(C, C^n)\}$.

The following is clear:

Proposition 2.28. *Every chain complex is trivially a coalgebra over its own coendomorphism operad.*

2.5. Examples

Example 2.29. Coassociative coalgebras are precisely the coalgebras over $\mathfrak{S}_0$ (see 2.21).

Definition 2.30. *Cocommut* is an operad defined to have one basis element $\{b_i\}$ for all integers $i \geqslant 0$. Here the rank of b_i is i and the degree is 0 and the these elements satisfy the composition-law: $\gamma(b_n \otimes b_{k_1} \otimes \cdots \otimes b_{k_n}) = b_K$, where $K = \sum_{i=1}^{n} k_i$. The differential of this operad is identically zero. The symmetric-group actions are trivial.

Example 2.31. Coassociative commutative coalgebras are the coalgebras over *Cocommut*.

The following example has many topological applications

Example 2.32. Coalgebras over the operad $\mathfrak{S}$, defined in 2.23, are chain-complexes equipped with a coassociative coproduct and Steenrod operations for all primes (see [8]).

3. The general construction

We begin by defining

Definition 3.1. Let $n \geqslant 1$ be an integer and let k be 0 or 1. Define $\mathcal{P}_k(n)$ to be the set of sequences $\{u_1, \dots, u_m\}$ of elements each of which is either a $\bullet$-symbol or an integer $\geqslant k$ and such that

$$\sum_{j=1}^{m} u_j = n, \tag{3.1}$$

where $\bullet = 1$ for the purpose of computing this sum.

Given a sequence $\mathbf{u} \in \mathcal{P}_k(n)$, let $|\mathbf{u}| = m$, the length of the sequence.

Remark 3.2. Note that the set $\mathcal{P}_1(n)$ is finite and for any $\mathbf{u} \in \mathcal{P}_k(n)$ $|\mathbf{u}| \leqslant n$. By contrast, $\mathcal{P}_0(n)$ is always infinite.

Definition 3.3. Let $\mathcal{V}$ be an operad, let C be a R-free DG module and set

$$k = \begin{cases} 0 & \text{if } \mathcal{V} \text{ is unital,} \\ 1 & \text{otherwise.} \end{cases}$$

Now define

$$KC = C \oplus \prod_{n \geqslant k} \operatorname{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr),$$

where $\operatorname{Hom}_{RS_0}(\mathcal{V}_0, C^0) = R$ in the unital case.

Consider the diagram

$$\begin{tikzcd} & \prod_{m \geqslant k} \operatorname{Hom}_{RS_m}(\mathcal{V}_m, (KC)^m) \arrow[d, hook, "y"] \\ KC \arrow[r, "0 \oplus \prod_{n \geqslant k} c_n"'] & \prod_{n \geqslant k} \prod_{\mathbf{u} \in \mathcal{P}_k(n)} \operatorname{Hom}_R(\mathcal{V}_{|\mathbf{u}|} \otimes \mathcal{V}_{u_1} \otimes \cdots \otimes \mathcal{V}_{u_{|\mathbf{u}|}}, C^n) \end{tikzcd} \tag{3.2}$$

where

(1) the c_n are defined by

$$c_n = \prod_{\mathbf{u} \in \mathcal{P}_k(n)} \operatorname{Hom}_R(\gamma_{\mathbf{u}}, 1)$$

with

$$\operatorname{Hom}_R(\gamma_{\mathbf{u}}, 1) : \operatorname{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr) \to \operatorname{Hom}_R\bigl(\mathcal{V}_{|\mathbf{u}|} \otimes \mathcal{V}_{u_1} \otimes \cdots \otimes \mathcal{V}_{u_{|\mathbf{u}|}}, C^n\bigr)$$

the dual of the generalized structure map

$$\gamma_{\mathbf{u}} : \mathcal{V}_{|\mathbf{u}|} \otimes \mathcal{V}_{u_1} \otimes \cdots \otimes \mathcal{V}_{u_{|\mathbf{u}|}} \to \mathcal{V}_n$$

from Definition 2.16. We assume that $\mathcal{V}_\bullet = R$ and $C^\bullet = C$ so that $\operatorname{Hom}_{RS_\bullet}(\mathcal{V}_\bullet, C^\bullet) = C$.

(2) $y = \prod_{m \geqslant k} y_m$ and the maps

$$y_m : \mathrm{Hom}_{RS_m}\big(\mathcal{V}_m, (KC)^m\big) \to \prod_{\substack{n \geqslant k \\ \mathbf{u} \in \mathcal{P}_k(n)}} \mathrm{Hom}_R\big(\mathcal{V}_{|\mathbf{u}|} \otimes \mathcal{V}_{u_1} \otimes \cdots \otimes \mathcal{V}_{u_{|\mathbf{u}|}}, C^n\big)$$

map the factor

$$\mathrm{Hom}_{RS_m}\left(\mathcal{V}_m, \bigotimes_{j=1}^{m} L_j\right) \subset \mathrm{Hom}_{RS_m}\big(\mathcal{V}_m, (KC)^m\big)$$

with $L_j = \mathrm{Hom}_{RS_{u_j}}(\mathcal{V}_{u_j}, C^{u_j})$ via the map induced by the associativity of the Hom and $\otimes$ functors.

A submodule $M \subseteq KC$ is called $\mathcal{V}$*-closed* if

$$\left(0 \oplus \prod_{n \geqslant k} c_n\right)(M) \subseteq y\left(\prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, M^n\big)\right).$$

Now we take stock of the terms in diagram (3.2).

Lemma 3.4. *Let $\mathcal{V}$ be an operad and let $k = 0$ if $\mathcal{V}$ is unital and 1 otherwise. Under the hypotheses of Definition 3.3, if C is a DG module over R, set*

$$L_1 C = KC,$$
$$L_n C = \left(0 \oplus \prod_{n \geqslant k} c_n\right)^{-1} \prod_{m \geqslant k} y_m\big(\mathrm{Hom}_{RS_m}\big(\mathcal{V}_m, (L_{n-1}C)^m\big)\big).$$

Then

$$L_{\mathcal{V}} C = \bigcap_{n=1}^{\infty} L_n C \tag{3.3}$$

—the maximal $\mathcal{V}$-closed submodule of KC (in the notation of Definition 3.3)—is a coalgebra over $\mathcal{V}$ with coproduct given by

$$g = y^{-1} \circ \left(0 \oplus \prod_{n \geqslant k} c_n\right) : L_{\mathcal{V}} C \to \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, (L_{\mathcal{V}} C)^n\big). \tag{3.4}$$

Remark 3.5. See Appendix A for the proof.

Lemma 3.6. *Let $\mathcal{V}$ be an operad and let $k = 0$ if $\mathcal{V}$ is unital and 1 otherwise. Given a coalgebra D over $\mathcal{V}$ with adjoint structure maps*

$$d_n : D \to \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, D^n\big)$$

any morphism of DG-modules

$$f : D \to C$$

induces a map

$$\hat{f} = f \oplus \prod_{n \geqslant k}^{\infty} \mathrm{Hom}_{RS_n}\bigl(1, f^n\bigr) \circ d_n : D \to C \oplus \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr)$$

whose image lies in

$$L_{\mathcal{V}} C \subseteq C \oplus \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr)$$

as defined in Lemma 3.4. *Furthermore, $\hat{f}$ is a morphism of $\mathcal{V}$-coalgebras.*

Remark 3.7. In the unital case, the augmentation $L_{\mathcal{V}} C \to R$ is induced by projection to the factor $\mathrm{Hom}_{RS_0}(\mathcal{V}_0, C^0) = R$.

Proof. We prove the claim when $C = D$ and use the functoriality of $L_{\mathcal{V}} C$ with respect to morphisms of C to conclude it in the general case. In this case $f = \mathrm{id} : D \to D$ and $\hat{f} = d$.

We claim that the diagram

$$\begin{tikzcd}
D \arrow[r, "d"] \arrow[d, "\hat{f}"'] \arrow[drr, "y \circ \prod_{n \geqslant k} \mathrm{Hom}_R(1, d^n) \circ d" description] & \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, D^n) \arrow[r, "\prod_{n \geqslant k} \mathrm{Hom}_R(1, \hat{f}^n)"] & \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, (L_{\mathcal{V}} D)^n) \arrow[d, "y"] \\
L_{\mathcal{V}} D \arrow[rr, "0 \oplus \prod_{n \geqslant k} c_n"'] & & \prod_{n \geqslant k\, \mathbf{u} \in \mathcal{P}_k(n)} \mathrm{Hom}_R(\mathcal{V}_{|\mathbf{u}|} \otimes \mathcal{V}_{u_1} \otimes \cdots \otimes \mathcal{V}_{u_{|\mathbf{u}|}}, C^n)
\end{tikzcd} \tag{3.5}$$

commutes, where c_n, y, and y_n are as defined in Definition 3.3 so that the lower row and right column are the same as diagram (3.2). Clearly, the upper sub-triangle of this diagram commutes since $\hat{f} = d$. On the other hand, the lower sub-triangle also clearly commutes by the definition of c_n and the fact that D is a $\mathcal{V}$-coalgebra. It follows that the entire diagram commutes. But this implies that $\mathrm{im}\,\hat{f} \subseteq \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, D^n) = KD$ (in the notation of Definition 3.3) satisfies the condition that

$$\Bigl(0 \oplus \prod_{n \geqslant k} c_n\Bigr)\bigl(\mathrm{im}\,\hat{f}\bigr) \subseteq y\Bigl(\prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\bigl(\mathcal{V}_n, \bigl(\mathrm{im}\,\hat{f}\bigr)^n\bigr)\Bigr)$$

so that $\mathrm{im}\,\hat{f}$ is $\mathcal{V}$-closed—see Definition 3.3. It follows that $\mathrm{im}\,\hat{f} \subseteq L_{\mathcal{V}} D \subseteq KD$ since $L_{\mathcal{V}} D$ is *maximal* with respect to this property (see Lemma 3.4).

This implies both of the statements of this lemma. □

Theorem 3.8. *Let D be a coalgebra over the operad $\mathcal{V}$ with adjoint structure maps*

$$d_n : D \to \mathrm{Hom}_{RS_n}\bigl(\mathcal{V}_n, D^n\bigr)$$

and let

$$f : D \to C$$

be any morphism of DG-modules. Then the coalgebra morphism

$$\hat{f}: D \to L_{\mathcal{V}} C$$

defined in Lemma 3.6 *is the unique coalgebra morphism that makes the diagram*

$$\begin{tikzcd} D \arrow[r, "\hat{f}"] \arrow[dr, "f"'] & L_{\mathcal{V}} C \arrow[d, "\varepsilon"] \\ & C \end{tikzcd} \tag{3.6}$$

commute. Consequently $L_{\mathcal{V}} C$ is the cofree coalgebra over $\mathcal{V}$ cogenerated by C. The cogeneration map (*see Definition* 1.1) $\varepsilon: L_{\mathcal{V}} C \to C$ *is projection to the first direct summand.*

Proof. Let $k = 0$ if $\mathcal{V}$ is unital and 1 otherwise. It is very easy to see that diagram (3.6) commutes with $\hat{f}$ as defined in Lemma 3.6. Suppose that

$$g: D \to L_{\mathcal{V}} C$$

is another coalgebra morphism that makes diagram (3.6) commute. We claim that g must coincide with $\hat{f}$. The component

$$\mathrm{Hom}_R(\gamma_{\{\bullet,\dots,\bullet\}}, 1): L_{\mathcal{V}} C \to \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, (L_{\mathcal{V}} C)^n\big)$$

isomorphically maps

$$\mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, C^n\big)$$

to the direct summand

$$\mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, C^n\big) \subset \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, (L_{\mathcal{V}} C)^n\big).$$

For g to be a coalgebra morphism, we *must* have (at least)

$$\mathrm{Hom}_R\big(1, g^n\big) \circ d_n = \mathrm{Hom}_R(\gamma_{\{\bullet,\dots,\bullet\}}, 1) \circ g,$$

for all $n \geqslant k$. This requirement, however, *forces* $g = \hat{f}$.

Lemma 3.6 and the argument above verify all of the conditions in Definition 1.1. $\square$

4. Special cases

4.1. Dimension restrictions

Now we address the issue of our cofree coalgebra extending into negative dimensions. We need the following definition first:

Definition 4.1. If E is a chain-complex, and t is an integer, let $E^{\rhd t}$ denote the chain-complex defined by

$$E_i^{\rhd t} = \begin{cases} 0 & \text{if } i \leqslant t, \\ \ker \partial_{t+1}: E_{t+1} \to E_t & \text{if } i = t+1, \\ E_i & \text{if } i > t+1. \end{cases}$$

Corollary 4.2. *If C is a chain-complex concentrated in nonnegative dimensions and $\mathcal{V}$ is an operad, then there exists a sub-$\mathcal{V}$-coalgebra*

$$M_{\mathcal{V}}C \subset L_{\mathcal{V}}C$$

such that

(1) *as a chain-complex, $M_{\mathcal{V}}C$ is concentrated in nonnegative dimensions,*
(2) *for any $\mathcal{V}$-coalgebra, D, concentrated in nonnegative dimensions, the image of the classifying map*

$$\hat{f}: D \to L_{\mathcal{V}}C$$

lies in $M_{\mathcal{V}}C \subset L_{\mathcal{V}}C$.

In addition, $M_{\mathcal{V}}C$ can be defined inductively as follows: *Let $M_0 = (L_{\mathcal{V}}C)^{\triangleright -1}$ (see* 4.1) *with structure map*

$$\delta_0 : M_0 \to \prod_{n>0} \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, (L_{\mathcal{V}}C)^n\big) = Q_{-1}.$$

Now define

$$M_{i+1} = \delta_i^{-1}\big(\delta_i(M_i) \cap Q_i\big) \subseteq \delta_i^{-1} Q_{i-1}$$

with structure map

$$\delta_{i+1} = \delta_i | M_{i+1} : M_{i+1} \to Q_i,$$

where

$$Q_i = \prod_{n>0} \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, M_i^n\big)^{\triangleright -1},$$

Then

$$M_{\mathcal{V}}C = \bigcap_{i=0}^{\infty} M_i.$$

Remark 4.3. Our definition of $M_{\mathcal{V}}C$ is simply that of the maximal sub-coalgebra of $L_{\mathcal{V}}C$ contained within $L_{\mathcal{V}}C^{\triangleright -1}$—i.e., the maximal sub-coalgebra in *nonnegative* dimensions.

4.2. The pointed irreducible case

We define the pointed irreducible coalgebras over an operad in a way that extends the conventional definition in [10]:

Definition 4.4. Given a coalgebra over a unital operad $\mathcal{V}$ with adjoint structure map

$$a_n : C \to \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, C^n\big)$$

an *element* $c \in C$ is called *group-like* if $a_n(c) = f_n(c^n)$ for all $n > 0$. Here $c^n \in C^n$ is the n-fold R-tensor product,

$$f_n = \operatorname{Hom}_R(\varepsilon_n, 1) : \operatorname{Hom}_R\bigl(R, C^n\bigr) = C^n \to \operatorname{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr)$$

and $\varepsilon_n : \mathcal{V}_n \to R$ is the augmentation (which exists by 2.8).

A coalgebra C over a unital operad $\mathcal{V}$ is called *pointed* if it has a *unique* group-like element (denoted 1), and *pointed irreducible* if the intersection of any two sub-coalgebras contains this unique group-like element.

Remark 4.5. Note that a group-like element generates a sub $\mathcal{V}$-coalgebra of C and must lie in dimension 0.

Although this definition seems contrived, it arises in "nature": The chain-complex of a pointed, simply-simply connected reduced simplicial set is pointed irreducible over the operad $\mathfrak{S}$. In this case, the operad action encodes the effect on the chain level of all Steenrod operations.

Note that our cofree coalgebra in Theorem 3.4 is pointed since it has the sub-coalgebra R. It is *not* irreducible since the *null* submodule, C (on which the coproduct vanishes identically), is a sub-coalgebra whose intersection with R is 0. We conclude that:

Lemma 4.6. *Let C be a pointed irreducible coalgebra over a unital operad $\mathcal{V}$. Then the adjoint structure map*

$$C \to \prod_{n \geqslant 0} \operatorname{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr)$$

is injective.

The existence of units of operads, and the associativity relations imply that

Lemma 4.7. *Let C be a coalgebra over an operad $\mathcal{V}$ with the property that the adjoint structure map*

$$\prod_{n \geqslant 1} a_n : C \to \prod_{n \geqslant 1} \operatorname{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr)$$

is injective. Then the adjoint structure map

$$a_1 : C \to \operatorname{Hom}_R(\mathcal{V}_1, C)$$

is naturally split by

$$\operatorname{Hom}_R(\eta_1, 1) : \operatorname{Hom}_R(\mathcal{V}_1, C) \to \operatorname{Hom}_R(R, C) = C$$

where $\eta_1 : R \to \mathcal{V}_1$ is the unit.

Remark 4.8. In general, the unit $\eta_1 \in \mathcal{V}$ maps under the structure map

$$s : \mathcal{V} \to \operatorname{CoEnd}(C)$$

to a unit of $\operatorname{im} s$—a *sub*-operad of $\operatorname{CoEnd}(C)$. We show that $s(\eta_1)$ is $1 : C \to C \in \operatorname{CoEnd}(C)_1$.

Proof. Consider the endomorphism

$$e = \mathrm{Hom}_R(\eta_1, C) \circ a_1 : C \to C$$

The operad identities imply that the diagram

$$\begin{tikzcd} C \arrow[r, "\prod_{n\geqslant 1} a_n"] \arrow[d, "e"'] & \prod_{n\geqslant 1} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^n) \\ C \arrow[ur, "\prod_{n\geqslant 1} a_n"'] & \end{tikzcd}$$

commutes since η_1 is a unit of the operad and $\mathrm{Hom}_R(\eta_1, C) \circ a_1$ must preserve the coproduct structure (acting, effectively, as the *identity map*).

It follows that $e^2 = e$ and that $\ker e \subseteq \ker \prod_{n\geqslant 1} a_n$. The hypotheses imply that $\ker e = 0$ and we claim that $e^2 = e \Rightarrow \mathrm{im}\, e = C$. Otherwise, suppose that $x \in C \setminus \mathrm{im}\, e$. Then $e(x - e(x)) = 0$ so $x - e(x) \in \ker e$, which is a contradiction. The conclusion follows. □

Proposition 4.9. *Let D be a pointed, irreducible coalgebra over a unital operad $\mathcal{V}$. Then the augmentation map*

$$\varepsilon : D \to R$$

is naturally split and any morphism of pointed, irreducible coalgebras

$$f : D_1 \to D_2$$

is of the form

$$1 \oplus \bar{f} : D_1 = R \oplus \ker \varepsilon_{D_1} \to D_2 = R \oplus \ker \varepsilon_{D_2},$$

where $\varepsilon_i : D_i \to R$, $i = 1, 2$, are the augmentations.

Proof. Definition 4.4 of the sub-coalgebra $R \subseteq D_i$ is stated in an invariant way, so that any coalgebra morphism must preserve it. □

Our result is:

Theorem 4.10. *If C is a chain-complex and $\mathcal{V}$ is a unital operad, define*

$$P_{\mathcal{V}} C = L_{\mathcal{V}} C / C$$

(*see Theorem* 3.4) *with the induced quotient structure map.*

Then $P_{\mathcal{V}} C$ is a pointed, irreducible coalgebra over $\mathcal{V}$. Given any pointed, irreducible coalgebra D over $\mathcal{V}$ with adjoint structure maps

$$d_n : D \to \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, D^n\big)$$

and augmentation

$$\varepsilon : D \to R$$

any morphism of DG-modules

$$f : \ker \varepsilon \to C$$

extends to a unique morphism of pointed, irreducible coalgebras over $\mathcal{V}$

$$1 \oplus \hat{f} : R \oplus \ker \varepsilon \to P_{\mathcal{V}} C,$$

where

$$\hat{f} = 1 \oplus \prod_{n=1}^{\infty} \mathrm{Hom}_{RS_n}\bigl(1, f^n\bigr) \circ d_n : D \to P_{\mathcal{V}} C,$$

If $p_C : P_{\mathcal{V}} C \to \mathrm{Hom}_R(\mathcal{V}_1, C)$ *is projection to the first factor, and*

$$\mathrm{Hom}_R(\eta_1, 1) : \mathrm{Hom}_R(\mathcal{V}_1, C) \to C$$

is the splitting map defined in 4.7, *then the diagram*

$$\begin{tikzcd} D \arrow[r, "\hat{f}"] \arrow[dr, "f"'] & P_{\mathcal{V}} C \arrow[d, "\mathrm{Hom}_R(\eta_1,1)\circ p_C"] \\ & C \end{tikzcd} \tag{4.1}$$

commutes.

In particular, $P_{\mathcal{V}} C$ *is the cofree pointed irreducible coalgebra over* $\mathcal{V}$ *with cogeneration map* $\mathrm{Hom}_R(\eta, 1) \circ p_C$ (*see Definition* 1.1).

Remark 4.11. Roughly speaking, $P_{\mathcal{V}} C$ is an analogue to the *Shuffle Coalgebra* defined in [10, Chapter 11]. With one extra condition on the operad $\mathcal{V}$, this becomes a generalization of the Shuffle Coalgebra.

Proof. Since the kernel of the structure map of D vanishes

$$\mathrm{im}\, \hat{f} \cap C = 0$$

so that $\mathrm{im}\, \hat{f}$ is mapped isomorphically by the projection $L_{\mathcal{V}} C \to P_{\mathcal{V}} C$.

It is first necessary to show that $\mathrm{Hom}_R(\eta_1, 1) \circ p_C : \mathrm{Hom}_R(\mathcal{V}_1, C) \to C$ can serve as the cogeneration map, i.e., that diagram (4.1) commutes.

This conclusion follows from the commutativity of the diagram

$$\begin{tikzcd} D \arrow[r, "d"] \arrow[rr, bend left, "\hat{f}"] \arrow[dr, bend right, "="'] & P_{\mathcal{V}} D \arrow[r, "P_{\mathcal{V}} f"] \arrow[d, "\mathrm{Hom}_R(\eta_1,1_D)\circ p_D"] & P_{\mathcal{V}} C \arrow[d, "\mathrm{Hom}_R(\eta_1,1_C)\circ p_C"] \\ & D \arrow[r, "f"'] & C \end{tikzcd}$$

where $d : D \to P_{\mathcal{V}} C$ is the canonical classifying map of D.

The upper (curved) triangle commutes by the definition of $\hat{f}$, the lower left triangle by the fact that $\mathrm{Hom}_R(\eta_1, 1)$ splits the classifying map. The lower right square commutes by functoriality of $P_{\mathcal{V}} *$.

We must also show that $P_{\mathcal{V}}C$ is pointed irreducible. The sub-coalgebra generated by $1 \in R = \mathrm{Hom}_{RS_0}(\mathcal{V}_0, C^0)$ is group-like.

Claim. *If $x \in P_{\mathcal{V}}C$ is an arbitrary element, its coproduct in* $\mathrm{Hom}_{RS_N}(\mathcal{V}, P_{\mathcal{V}}C^N)$ *for N sufficiently large, contains factors of* $1 \in R \subset P_{\mathcal{V}}C$.

This follows from the fact that $\mathbf{u} \in \mathcal{P}_0(n)$ *must* have terms $u_i = 0$ for $N = |\mathbf{u}| > n$—see 3.1 with $k = 0$.

It follows that *every* sub-coalgebra of $P_{\mathcal{V}}C$ must contain 1 so that R is the *unique* sub-coalgebra of $P_{\mathcal{V}}C$ generated by a group-like element. This implies that $P_{\mathcal{V}}C$ is pointed irreducible.

The statement about any pointed irreducible coalgebra mapping to $P_{\mathcal{V}}C$ follows from Lemma 3.6. $\square$

Definition 4.12. Let C be a pointed irreducible $\mathcal{V}$-coalgebra with augmentation

$$\varepsilon : C \to R.$$

If t is some integer, we say that C is *t-reduced* if

$$(\ker \varepsilon)_i = 0$$

for all $i \leqslant t$.

Remark 4.13. If $t \geqslant 1$, the chain complex of a t-reduced simplicial set (see [4, p. 170]) is naturally a t-reduced pointed, irreducible coalgebra over $\mathfrak{S}$. The case where $t \leqslant 0$ also occurs in topology in the study of spectra.

We conclude this section with a variation of 4.2.

Proposition 4.14. *If t is an integer and C is a chain-complex concentrated in dimensions $> t$, and $\mathcal{V}$ is a unital operad, let $P_{\mathcal{V}}C$ be the pointed, irreducible coalgebra over $\mathcal{V}$ defined in* 4.10. *There exists a sub-coalgebra,*

$$\mathcal{F}_{\mathcal{V}}^{\triangleright t} C \subset P_{\mathcal{V}}C$$

such that

(1) *$\mathcal{F}_{\mathcal{V}}^{\triangleright t} C$ is a t-reduced pointed irreducible coalgebra over $\mathcal{V}$,*
(2) *for any pointed, irreducible t-reduced $\mathcal{V}$-coalgebra, D, the image of the classifying map*

$$1 \oplus \hat{f} : D \to P_{\mathcal{V}}C$$

lies in $\mathcal{F}_{\mathcal{V}}^{\triangleright t} C \subset P_{\mathcal{V}}C$.

In addition, $\mathcal{F}_{\mathcal{V}}^{\triangleright t} C$ can be defined inductively as follows: Let $Y_0 = R \oplus (P_{\mathcal{V}}C)^{\triangleright t}$ (see 4.1 *for the definition of $(*)^{\triangleright t}$) with structure map*

$$\alpha_0 : Y_0 \to R \oplus \prod_{n>0} \mathrm{Hom}_{RS_n}\bigl(\mathcal{V}_n, (P_{\mathcal{V}}C)^n\bigr) = Z_{-1}.$$

Now define

$$Y_{i+1} = \alpha_i^{-1}\big(\alpha_i(Y_i) \cap Z_i\big) \subseteq \alpha_i^{-1} Z_{i-1} \tag{4.2}$$

with structure map

$$\alpha_{i+1} = \alpha_i | Y_{i+1} : Y_{i+1} \to Z_i,$$

where

$$Z_i = R \oplus \prod_{n>0} \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, Y_i^n\big)^{\rhd t}.$$

Then

$$\mathcal{F}_{\mathcal{V}}^{\rhd t} C = \bigcap_{i=0}^{\infty} Y_i.$$

Remark 4.15. Our definition of $\mathcal{F}_{\mathcal{V}}^{\rhd t} C$ is simply that of the maximal sub-coalgebra of $P_{\mathcal{V}} C$ contained within $R \oplus P_{\mathcal{V}} C^{\rhd t}$.

Example 4.16. For example, let $\mathcal{V} = \mathfrak{S}_0$—the operad whose coalgebras are coassociative coalgebras. Let C be a chain-complex concentrated in positive dimensions. Since the operad is concentrated in dimension 0 the "natural" coproduct given in 4.10 does *not* go into negative dimensions when applied to $R \oplus \prod_{n>0} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^n)^{\rhd 0}$ so $M_n C = \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^n)^{\rhd 0} = \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^n)$ for all $n > 0$ and

$$\begin{aligned}
\mathcal{F}_{\mathcal{V}}^{\rhd 0} C &= R \oplus \prod_{n>0} \mathrm{Hom}_{RS_n}\big(\mathcal{V}_n, C^n\big)^{\rhd 0} \\
&= R \oplus \bigoplus_{n>0} \mathrm{Hom}_{RS_n}\big(n, C^n\big) \\
&= T(C)
\end{aligned}$$

the *tensor-algebra*—the well-known pointed, irreducible cofree coalgebra used in the *bar construction.*

The fact that the direct product is of *graded* modules and dimension considerations imply that, in each dimension, it only has a *finite* number of nonzero factors. So, in this case, the direct product becomes a direct sum.

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Appendix A. Proof of Lemma 3.4

As always, $k = 0$ if $\mathcal{V}$ is unital and 1, otherwise. Note that the coproduct formula, Eq. (3.4) is well-defined because the map

$$y = \prod_{m \geqslant k} y_m$$

is injective and

$$\Bigl(0 \oplus \prod_{n \geqslant k} c_n\Bigr)(L_{\mathcal{V}} C) \subseteq y\Bigl(\prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\bigl(\mathcal{V}_n, (L_{\mathcal{V}} C)^n\bigr)\Bigr)$$

by our construction of $L_{\mathcal{V}} C$ in Eq. (3.3).

The basic idea behind this proof is that we *dualize* the argument used in verifying the defining properties of a free algebra over an operad in [6]. This is complicated by the fact that $L_{\mathcal{V}} C$ is not really the dual of a free algebra. The closest thing we have to this dual is KC in Definition 3.3. But $L_{\mathcal{V}} C$ is *contained* in KC, not equal to it. We cannot dualize the proof that a free $\mathcal{V}$-algebra is free, but can carry out a similar argument with respect to a kind of "Hilbert basis" of $L_{\mathcal{V}} C$.

Consider a factor

$$\mathrm{Hom}_{RS_\alpha}\bigl(\mathcal{V}_\alpha, C^\alpha\bigr) \subset C \oplus \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr).$$

In general

$$\mathrm{Hom}_{RS_\alpha}\bigl(\mathcal{V}_\alpha, C^\alpha\bigr) \not\subset L_{\mathcal{V}} C \subset C \oplus \prod_{n \geqslant k} \mathrm{Hom}_{RS_n}\bigl(\mathcal{V}_n, C^n\bigr)$$

but we still have a projection

$$p_\alpha : L_{\mathcal{V}} C \to \mathrm{Hom}_{RS_\alpha}\bigl(\mathcal{V}_\alpha, C^\alpha\bigr).$$

Let its image be $K_\alpha \subseteq \mathrm{Hom}_{RS_\alpha}(\mathcal{V}_\alpha, C^\alpha)$. We will show that all faces of the diagram in Fig. A.1 other than the front face commute for all α, n, m and $\mathbf{u} \in \mathcal{P}_k(\alpha)$, with $u_i \neq \bullet$ and $|\mathbf{u}| = n$, $\mathbf{v} \in \mathcal{P}_k(u_i)$ with $|\mathbf{v}| = m$ and $\mathbf{w} \in \mathcal{P}_k(\alpha)$ where $\mathbf{w}$ is the result of replacing the ith entry of $\mathbf{u}$ by $\mathbf{v}$, so coproduct on the copy of C represente by $u_i = \bullet$ would vanish. Here, ι is the composite

$$\begin{array}{c}
\mathrm{Hom}_{RS_n}(\mathcal{V}_n, L_{\mathcal{V}} C^{i-1} \otimes \mathrm{Hom}_{RS_m}(\mathcal{V}_m, L_{\mathcal{V}} C^m) \otimes L_{\mathcal{V}} C^{n-i}) \\
\Big\| \\
\mathrm{Hom}_{RS_n}(\mathcal{V}_n, \mathrm{Hom}_R(R, L_{\mathcal{V}} C^{i-1}) \otimes \mathrm{Hom}_{RS_m}(\mathcal{V}_m, L_{\mathcal{V}} C^m) \otimes \mathrm{Hom}_R(R, L_{\mathcal{V}} C^{n-i})) \\
\Big\downarrow \\
\mathrm{Hom}_R(\mathcal{V}_n, \mathrm{Hom}_R(\mathcal{V}_m, L_{\mathcal{V}} C^{n+m-1})) \\
\Big\downarrow \\
\mathrm{Hom}_R(\mathcal{V}_n \otimes \mathcal{V}_m, L_{\mathcal{V}} C^{n+m-1})
\end{array} \tag{A.1}$$

$$
\begin{tikzcd}
 & K_\alpha \arrow[rr, "y_n^{-1}\circ \mathrm{Hom}_R(\gamma_{\mathbf{u}},1)"] \arrow[dd, "y_{n+m-1}^{-1}\circ \mathrm{Hom}_R(\gamma_{\mathbf{w}},1)"] & & \mathrm{Hom}_{RS_n}(\mathcal{V}_n, \bigotimes_{j=1}^{n} K_{u_j}) \arrow[dd, "\iota_1\circ \mathrm{Hom}_R(1,1^{i-1}\otimes(y_m^{-1}\mathrm{Hom}_R(\gamma_{\mathbf{V}},1))\otimes 1^{n-i})"] \\
L_{\mathcal{V}}C \arrow[ur, "p_\alpha"] \arrow[rr, "g_n"] \arrow[dd, "g_{n+m-1}"'] & & \mathrm{Hom}_{RS_n}(\mathcal{V}_n, (L_{\mathcal{V}}C)^n) \arrow[ur, "\mathrm{Hom}_{RS_n}(1,\bigotimes_{j=1}^{n} p_{u_j})"] \arrow[dd, "\iota\circ \mathrm{Hom}_R(1,\underbrace{1\otimes\cdots\otimes g_m\otimes\cdots\otimes 1}_{i\text{th position}})"] & \\
 & \mathrm{Hom}_{RS_{n+m-1}}(\mathcal{V}_{n+m-1}, \bigotimes_{j=1}^{n+m-1} K_{w_j}) \arrow[rr, "\mathrm{Hom}_R(\circ_i,1)"'] & & \mathrm{Hom}_R(\mathcal{V}_n\otimes\mathcal{V}_m, \bigotimes_{j=1}^{n+m-1} K_{w_j}) \\
\mathrm{Hom}_{RS_{n+m-1}}(\mathcal{V}_{n+m-1}, (L_{\mathcal{V}}C)^{n+m-1}) \arrow[ur, "\mathrm{Hom}_R(1,\bigotimes_{j=1}^{n+m-1} p_{w_j})"] \arrow[rr, "\mathrm{Hom}_R(\circ_i,1)"'] & & \mathrm{Hom}_R(\mathcal{V}_n\otimes\mathcal{V}_m, (L_{\mathcal{V}}C)^{n+m-1}) \arrow[ur, "\mathrm{Hom}_R(1\otimes 1,\bigotimes_{j=1}^{n+m-1} p_{w_j})"'] &
\end{tikzcd}
$$

Fig. A.1.

and ι_1 is the composite

$$\begin{array}{c}
\mathrm{Hom}_{RS_n}(\mathcal{V}_n, \bigotimes_{j=1}^{i-1} K_{u_j} \otimes \mathrm{Hom}_{RS_m}(\mathcal{V}_m, \bigotimes_{j=1}^{m} K_{v_j}) \otimes \bigotimes_{j=i+1}^{n} K_{u_j}) \\
\Vert \\
\mathrm{Hom}_{RS_n}(\mathcal{V}_n, A \otimes \mathrm{Hom}_{RS_m}(\mathcal{V}_m, \bigotimes_{j=1}^{m} K_{v_j}) \otimes B) \\
\downarrow \\
\mathrm{Hom}_{R}(\mathcal{V}_n, \mathrm{Hom}_{R}(\mathcal{V}_m, \bigotimes_{j=1}^{n+m-1} K_{w_j})) \\
\downarrow \\
\mathrm{Hom}_{R}(\mathcal{V}_n \otimes \mathcal{V}_m, \bigotimes_{j=1}^{n+m-1} K_{w_j})
\end{array} \tag{A.2}$$

with $A = \bigotimes_{j=1}^{i-1} \mathrm{Hom}_R(R, K_{u_k})$, $B = \bigotimes_{j=i+1}^{n} \mathrm{Hom}_R(R, K_{u_k})$. The top face commutes by the definition of the coproduct of $L_{\mathcal{V}}C$ and the fact that the image of the coproduct of an element $x \in L_{\mathcal{V}}C$ under $\mathrm{Hom}_R(1, \bigotimes_{k=1}^{n} p_{u_k})$ *only* depends on $p_\alpha(x)$—since $\sum_{k=1}^{n} u_k = \alpha$. This also implies that the left face commutes since the left face is the same as the top face (for g_{n+m-1} rather than g_n).

To see that the right face commutes, note that ι and ι_1 are very similar—each term of diagram (A.2) projects to the corresponding term of diagram (A.1). The naturality of the projection maps and the fact that the top face of the diagram in Fig. A.1 commutes implies that the right face commutes.

Note that Definition 3.1 implies that

$$\alpha = \sum_{j=1}^{n} u_j = \sum_{j=1}^{n+m-1} w_j,$$
$$u_i = \sum_{j=1}^{m} v_j.$$

Since elements of K_α are determined by their projections, the commutativity of all faces of the diagram in Fig. A.1 except the front also implies that the *front* face commutes. This will prove Lemma 3.4 since it implies that diagram (2.7) of Definition 2.26 commutes.

The bottom face of the diagram in Fig. A.1 commutes by the functoriality of $\mathrm{Hom}_R(*, *)$.

It remains to prove that the *back* face commutes. To establish this, we consider the diagram in Fig. A.2, where

$$\begin{aligned}
s: \mathcal{V}_n \otimes \mathcal{V}_m \otimes \bigotimes_{j=1}^{n+m-1} \mathcal{V}_{w_j} &= \mathcal{V}_n \otimes \mathcal{V}_m \otimes \bigotimes_{j=1}^{i-1} \mathcal{V}_{u_j} \otimes \left(\bigotimes_{\ell=1}^{u_i} \mathcal{V}_{v_\ell} \right) \otimes \bigotimes_{j=i+1}^{n} \mathcal{V}_{u_j} \\
&\to \mathcal{V}_n \otimes \bigotimes_{j=1}^{i-1} \mathcal{V}_{u_j} \otimes \left(\mathcal{V}_m \otimes \bigotimes_{\ell=1}^{u_i} \mathcal{V}_{v_\ell} \right) \otimes \bigotimes_{j=i+1}^{n} \mathcal{V}_{u_j}
\end{aligned}$$

$$\begin{tikzcd}[column sep=small]
& \mathrm{Hom}_{RS_\alpha}(\mathcal{V}_\alpha, C^\alpha) \arrow[rr, "\mathrm{Hom}_R(\gamma_{\mathbf{u}},1)"] \arrow[dd, "\mathrm{Hom}_R(\gamma_{\mathbf{w}},1)"] & & \mathrm{Hom}_R(\mathcal{V}_n \otimes \bigotimes_{j=1}^{n} \mathcal{V}_{u_j}, C^\alpha) \arrow[dd, "\mathrm{Hom}_R(s\circ 1\otimes 1^{i-1}\otimes\gamma_{\mathbf{v}}\otimes 1^{n-i},1)"] \\
K_\alpha \arrow[ur, hook] \arrow[rr, "y_n^{-1}\circ\mathrm{Hom}_R(\gamma_{\mathbf{u}},1)"] \arrow[dd, "y_{n+m-1}^{-1}\circ\mathrm{Hom}_R(\gamma_{\mathbf{w}},1)"'] & & \mathrm{Hom}_{RS_n}(\mathcal{V}_n, \bigotimes_{j=1}^{n} K_{u_j}) \arrow[ur, hook, "y_n"] \arrow[dd, "\iota_1\circ\mathrm{Hom}_R(1,1^{i-1}\otimes(y_m^{-1}\mathrm{Hom}_R(\gamma_{\mathbf{v}},1))\otimes 1^{n-i})"] & \\
& \mathrm{Hom}_R(\mathcal{V}_{n+m-1} \otimes \bigotimes_{j=1}^{n+m-1} \mathcal{V}_{w_j}, C^\alpha) \arrow[rr, "\mathrm{Hom}_R(\circ_i\otimes 1,1)"'] & & \mathrm{Hom}_R(\mathcal{V}_n \otimes \mathcal{V}_m \otimes \bigotimes_{j=1}^{n+m-1} \mathcal{V}_{w_j}, C^\alpha) \\
\mathrm{Hom}_{RS_{n+m-1}}(\mathcal{V}_{n+m-1}, \bigotimes_{j=1}^{n+m-1} K_{w_j}) \arrow[ur, hook, "y_{n+m-1}"] \arrow[rr, "\mathrm{Hom}_R(\circ_i,1)"'] & & \mathrm{Hom}_R(\mathcal{V}_n \otimes \mathcal{V}_m, \bigotimes_{j=1}^{n+m-1} K_{w_j}) \arrow[ur, hook, "\iota_2"'] &
\end{tikzcd}$$

Fig. A.2.

is the shuffle map and ι_2 is the composite

$$\begin{tikzcd}
\mathrm{Hom}_R(\mathcal{V}_n \otimes \mathcal{V}_m, \bigotimes_{j=1}^{n+m-1} K_{w_j}) \arrow[d, hook, "\mathrm{Hom}_R(1, \bigotimes_{j=1}^{n+m-1} y_{w_j})"] \\
\mathrm{Hom}_R(\mathcal{V}_n \otimes \mathcal{V}_m, \bigotimes_{j=1}^{n+m-1} \mathrm{Hom}_{RS_{w_j}}(\mathcal{V}_{w_j}, C^{w_j})) \arrow[d] \\
\mathrm{Hom}_R(\mathcal{V}_n \otimes \mathcal{V}_m, \mathrm{Hom}_R(\bigotimes_{j=1}^{n+m-1} \mathcal{V}_{w_j}, C^{\alpha})) \arrow[d] \\
\mathrm{Hom}_R(\mathcal{V}_n \otimes \mathcal{V}_m \otimes \bigotimes_{j=1}^{n+m-1} \mathcal{V}_{w_j}, C^{\alpha})
\end{tikzcd}$$

where the maps in the lower two rows are the natural associativity maps for the Hom_R-functor and $\otimes$.

Clearly, the left and top faces of the diagram in Fig. A.2 commute. The *bottom* face also commutes because

(1) the maps $\mathrm{Hom}_R(\circ_i, 1)$ and $\mathrm{Hom}_R(\circ_i \otimes 1, 1)$ only affect the first argument in the $\mathrm{Hom}_R(*, *)$-functor and the other maps in the bottom face only affect the second (so there is no interactions between them)
(2) the remaining maps in that face are composites of natural multilinear associativity maps like those listed in Eqs. (B.1)–(B.4), so they commute by Theorem B.9.

The *rear* face commutes because the diagram

$$\begin{tikzcd}
\mathcal{V}_\alpha & & \mathcal{V}_n \otimes \bigotimes_{k=1}^{n} \mathcal{V}_{u_k} \arrow[ll, "\gamma_{\mathbf{u}}"'] \\
\mathcal{V}_{n+m-1} \otimes \bigotimes_{k=1}^{n+m-1} \mathcal{V}_{w_k} \arrow[u, "\gamma_{\mathbf{w}}"] & & \mathcal{V}_n \otimes \mathcal{V}_m \otimes \bigotimes_{k=1}^{n+m-1} \mathcal{V}_{w_k} \arrow[ll, "\circ_i \otimes 1"] \arrow[u, "1^{i-1} \otimes \gamma_{\mathbf{v}} \otimes 1^{n-i} \circ s"']
\end{tikzcd}$$

commutes due to the associativity relations for an operad—see Lemma 2.18.

It remains to prove that the *right* face of the diagram in Fig. A.2 commutes. We note that all of the morphisms involved in the right face are of the type listed in Eqs. (B.1)–(B.4) except for $\gamma_{\mathbf{v}}$ and invoke Theorem B.14

Appendix B. Multilinear functors

In this appendix, we consider multilinear functors on the category of free R-modules and show that certain natural transformations of them must be canonically equal.

Definition B.1. An *expression tree* is a rooted, ordered tree whose nodes are labeled with symbols Hom and $\otimes$ such that

(1) every node labeled with Hom has precisely two children,
(2) every node labeled with $\otimes$ can have an arbitrary (finite) number of children,
(3) leaf nodes are labeled with *distinct* R-modules.

Nodes are assigned a quality called *variance* (*covariance* or *contravariance*) as follows:

(1) The root is covariant.
(2) All children of a $\otimes$-node and the right child of a Hom-node have the *same* variance as it.
(3) The left child of a Hom-node is given the *opposite* variance.

Two expression-trees are regarded as the *same* if there exists an isomorphism of ordered trees between them that preserves node-labels.

Remark B.2. For instance, Fig. B.1 is an expression tree. That expression-trees are *rooted and ordered* means that:

(1) there is a distinguished node called the *root* that is preserved by isomorphisms;
(2) the children of every interior node have a well-defined *ordering* that is preserved by any isomorphism.

Definition B.3. Given an expression tree T, let $M(T)$ denote the R-module defined recursively by the rules

(1) if T is a single leaf-node labeled by a R-module A, then $M(T) = A$;

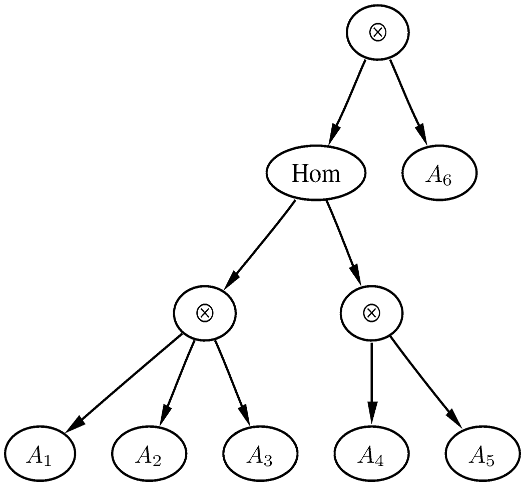

Fig. B.1.

(2) if the root of T is labeled with Hom and its two children are expression-trees T_1 and T_2, respectively, then

$$M(T) = \mathrm{Hom}_R\big(M(T_1), M(T_2)\big);$$

(3) if the root of T is labeled with $\otimes$ and its children are expression-trees $T_1, \dots, T_n$ then

$$M(T) = \bigotimes_{i=1}^{n} M(T_i).$$

Remark B.4. This associates a multilinear functor of the leaf-nodes with an expression tree.

For instance, if T is the expression tree in Remark B.2, then

$$M(T) = \mathrm{Hom}_R(A_1 \otimes A_2 \otimes A_3, A_4 \otimes A_5) \otimes A_6.$$

In other words, T is nothing but the *syntax tree* of the functors that make up $M(T)$.

Now we define *operations* that can be performed on expression trees and their effect on the associated functors.

Throughout this discussion, T is some fixed expression tree.

Definition B.5. *Type-*0 *transformations.* Perform the following operations or their inverses:

Hom-*transform*: Given any subtree, A of T, replace it by the subtree in Fig. B.2.
$\otimes$-*transform*: Given a subtree of the form of Fig. B.3, where $n > 0$ is some integer and $T_1, \dots, T_n$ are subtrees, replace it by Fig. B.4,

where $0 \leqslant i \leqslant n$.

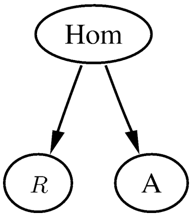

Fig. B.2.

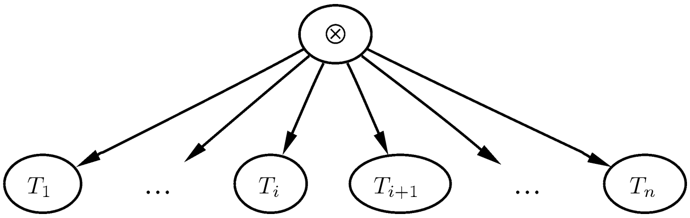

Fig. B.3.

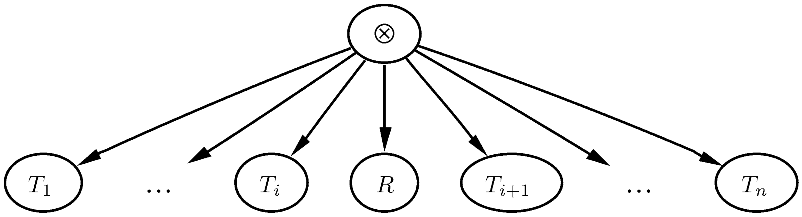

Fig. B.4.

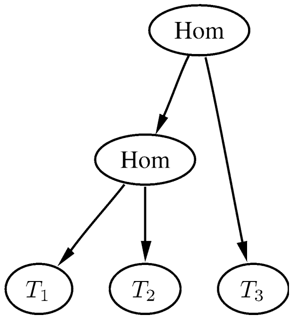

Fig. B.5.

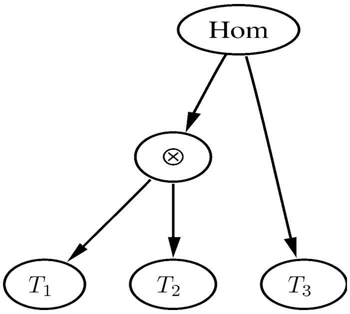

Fig. B.6.

In addition, we define slightly more complex transformations

Definition B.6. *Type*-1 *transformations*. Perform the following operation or its inverse: If T has a *covariant* node that is the root of a subtree like in Fig. B.5, where T_1, T_2, and T_3 are subtrees, replace it by the subtree in Fig. B.6.

If it has a *contravariant* node that is the root of a subtree like Fig. B.6, replace it by the subtree depicted in Fig. B.5.

Finally, we define the most complex transformation of all

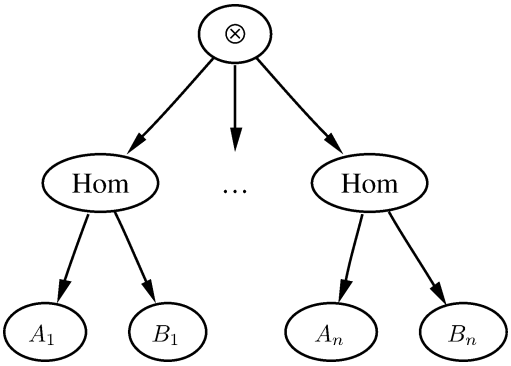

Fig. B.7.

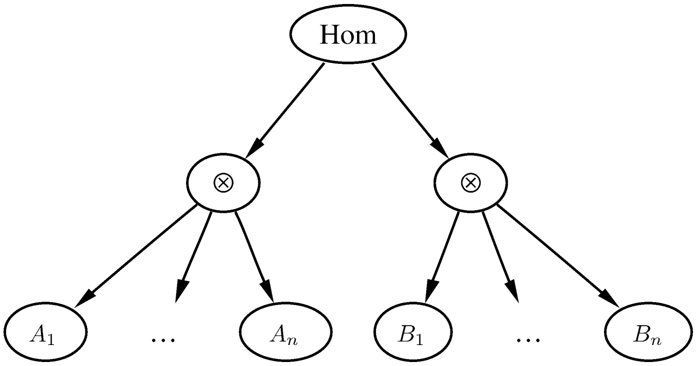

Fig. B.8.

Definition B.7. *Type*-2 *transformations*. If T is an expression tree with a *covariant* node that is the root of this subtree like in Fig. B.7, where $n > 1$ is an integer and $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are subtrees, we replace the subtree in Fig. B.7 by Fig. B.8.

If a node is *contravariant* and is the root of a subtree like Fig. B.8, we replace it by the tree in Fig. B.7.

Given these rules for transforming an expression tree, we can define an *induced natural transformation* of functors $M(T)$:

Claim B.8. *Let T be an expression tree and let T' be the result of performing a transform e, defined above, on T. Then there exists an induced natural transformation of functors*

$$f(e): M(T) \to M\big(T'\big).$$

Given a sequence $\mathbf{E} = \{e_1, \ldots, e_k\}$ of elementary transforms, we define $f(\mathbf{E})$ to be the composite of the $f(e_i)$, $i = 1, \ldots, n$.

This claim follows immediately from the recursive description of $M(T)$ in Definition B.3, the well-known morphisms

$$\mathrm{Hom}_R(R, A) = A, \tag{B.1}$$

$$A \otimes R \otimes B = A \otimes B, \tag{B.2}$$

$$\mathrm{Hom}_R\big(A, \mathrm{Hom}_R(B, C)\big) = \mathrm{Hom}_R(A \otimes B, C), \tag{B.3}$$

$$\mathrm{Hom}_R(A, B) \otimes \mathrm{Hom}_R(C, D) \to \mathrm{Hom}_R(A \otimes C, B \otimes D) \tag{B.4}$$

(where A, B, C, and D are free R-modules), and the functoriality of $\otimes$ and $\mathrm{Hom}_R(*, *)$.

In the case where the R-modules are DG-modules, we apply the Koszul convention for type-2 transformations such a transformation sends

$$(a \mapsto b) \otimes (c \mapsto d)$$

to

$$(-1)^{\dim b \cdot \dim c} a \otimes c \mapsto b \otimes d.$$

The Koszul conventions does *not* produce a change of sign in any of the other cases.

Now we are ready to state the main result of the appendix:

Theorem B.9. *Let T be an expression tree and suppose $\mathbf{E}_1$ and $\mathbf{E}_2$ are two sequences of elementary transformations* (*as defined in Definitions* B.5 *through* B.7) *that both result in the same transformed tree, T'. Then*

$$f(\mathbf{E}_1) = f(\mathbf{E}_2) : M(T) \to M\big(T'\big).$$

This result remains true if the R-free modules on the leaves are DG-modules and we follow the Koszul Convention.

Remark B.10. “Same” in this context means “isomorphic.” This theorem shows that the induced natural transformation, $f(\mathbf{E})$, *only* depends on the structure of the resulting tree, not on the sequence of transforms used. There is *less structure* to maps of the form $f(\mathbf{E})$ than one might think.

We devote the rest of this section to proving this result. We begin with

Definition B.11. Let T be an expression tree. Then inorder(T) denote the list of leaf-nodes of T as encountered in an in-order traversal of T, i.e.,

(1) if T is a single node A, then $\mathrm{inorder}(T) = \{A\}$
(2) if the root of T has child-subtrees $A_1, \ldots, A_n$ then

$$\mathrm{inorder}(T) = \mathrm{inorder}\ (A_1) \bullet \cdots \bullet\ \mathrm{inorder}\ (A_n),$$

where $\bullet$ denotes concatenation of lists.

Given transformations and in-order traversals, we want to record the effect of the transformations on these ordered lists.

Proposition B.12. *Let T be an expression tree and suppose the R-free modules on its leaves are equipped with R-bases. Then an element $x \in M(T)$ can be described as a set of lists*

$$x = \{(a_1, \dots, a_k) \dots\},$$

*where $a_i \in A_i$ and A_i is the free R-module occurring in the i*th *node in* inorder(T).

Remark B.13. To actually *define* $M(T)$ as a free R-module, we must add quantifiers and relations that depend on the internal structure of T to these lists.

Proof. Let A and B be free R-modules. Elements of $A \otimes B$ can be described as $a \otimes b$, where $a \in A$, $b \in B$ are basis elements. So the list in this case has two elements and the set of lists contains a single element:

$$\{(a, b)\}.$$

Elements of $\mathrm{Hom}_R(A, B)$ are functions from A to B—i.e., a set of ordered pairs

$$\{(a_1, b_1), \dots, (a_i, b_i, \dots)\},$$

where $a_i \in A$ is a basis element, $b_i \in B$ (not necessarily a basis element) and *every* basis element of A occurs as the left member of some ordered pair. The general statement follows from the recursive definition of $M(T)$ in Definition B.3 and the definition of in-order traversal in Definition B.11.

Now we prove Theorem B.9:

Let $x \in M(T)$ be given by

$$x = \{(a_1, \dots, a_k) \dots\}$$

as in Proposition B.12. We consider the effect of the transformations defined in Definitions B.5–B.7 on this element.

Type-0: transformations insert or remove terms equal to $1 \in R$ into each list in the set.

Type-1: transformations have *no effect* on the lists (they only affect the *predicates* used to define the module whose elements the lists represent).

Type-2: transformations permute portions of each list in x. In the DG case, whenever an element a is permuted past an element b, the list is multiplied by $(-1)^{\dim a \cdot \dim b}$.

Note that, in *no* case is the *data* in the lists altered. Furthermore, we claim that the equality of the trees resulting from performing $\mathbf{E}_1$ and $\mathbf{E}_2$ on T implies that:

- the permutations of the lists from the type-2 transformations must be compatible;
- the copies of R inserted or removed by the type-0 transformations must be in compatible locations on the tree.

Consequently, the lists that result from performing $\mathbf{E}_1$ and $\mathbf{E}_2$ on the lists of x must be the same and

$$f(\mathbf{E}_1)(x) = f(\mathbf{E}_2)(x).$$

The isomorphism of final expression trees also implies that the predicates that apply to corresponding element of these lists are also the same. Since this is true for an *arbitrary* x we conclude that

$$f(\mathbf{E}_1) = f(\mathbf{E}_2).$$

In the *DG case,* we note that type-2 transformations may introduce a change of sign. Nevertheless, the fact that the elements in the lists are in the same order implies that they have been permuted in the same way—and therefore have the same sign-factor. $\square$

We can generalize (relativize) Theorem B.9 slightly. We get a result like Theorem B.9 except that we have introduced a morphism that is *not* of the type

Theorem B.14. *Let T be an expression tree whose leaf-modules are $\{A_1, \ldots, A_n\}$ and consider the diagram*

$$\begin{tikzcd}
& T_1 \arrow[r, "\varphi"] & T_3 \arrow[dr, "\mathbf{E}_3"] & \\
T \arrow[ur, "\mathbf{E}_1"] \arrow[dr, "\mathbf{E}_2"'] & & & T' \\
& T_2 \arrow[r, "\varphi"'] & T_4 \arrow[ur, "\mathbf{E}_4"'] &
\end{tikzcd}$$

where

(1) *for some fixed index k, $\varphi : A_k \to \overline{T}$ replaces the leaf node labeled with the module, A_k, with an expression tree $\overline{T}$ that has leaf-modules $\{B_1, \ldots, B_t\}$;*
(2) $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, *and* $\mathbf{E}_4$ *are sequences of elementary transformations* (*as defined in Definitions* B.5 *through* B.7);
(3) *$f(\varphi) : A_k \to M(\overline{T})$ is some morphism of free R-modules.*

Then

$$f(\mathbf{E}_3) \circ f(\varphi) \circ f(\mathbf{E}_1) = f(\mathbf{E}_4) \circ f(\varphi) \circ f(\mathbf{E}_2).$$

Remark B.15. In other words, Theorem B.9 is still true if we have a morphism in the mix that is *not* of the canonical type in Eqs. (B.1)–(B.4)—as long as the remaining transformations are done in a compatible way.

Proof. Let $x \in M(T)$ be given by

$$x = \big\{(a_1, \ldots, a_n) \ldots\big\}.$$

We get

$$f(\mathbf{E}_1)(x) = \big\{(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \ldots\big\},$$
$$f(\mathbf{E}_2)(x) = \big\{(a_{\tau(1)}, \ldots, a_{\tau(n)}) \ldots\big\},$$

where $\sigma, \tau \in S_n$ are permutations. In each of these lists, we replace a_k by a set of lists

$$\{(b_1, \ldots, b_t)\}$$

representing the value of $\varphi(a_k)$ and apply $f(\mathbf{E}_3)$ and $f(\mathbf{E}_4)$, respectively—possibly permuting the resulting longer lists. As in Theorem B.9, the result is two copies of the same set of lists. This is because both sets of operations result in the expression tree T', implying that the permutations must be compatible. As in Theorem B.9, the key fact is that the data in the lists is not changed (except for being permuted). $\square$

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